The method of characteristics for steady supersonic rotational flow in three dimensions

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SUMMARY

The method of characteristics for steady supersonic flow problems in three dimensions, due to Coburn & Dolph (1949), is extended so that flow with shocks and entropy changes may be treated. Equations of motion based on Coburn & Dolph's characteristic coordinate system are derived and a scheme is described for solving these by finite differences.

A linearized method of characteristics is developed for calculating perturbations of a given three-dimensional field of flow. This is a generalization of the method evolved by Ferri (1952) for perturbations of plane flow and conical flow.

1. INTRODUCTION

The method of characteristics for problems of steady supersonic flow in two independent variables has a sound and clear cut basis which appears to be lacking when additional independent variables are considered. In two dimensions two distinct families of characteristic curves are defined uniquely and the independent variables can be transformed so that the problem expressed in terms of characteristic coordinates is equivalent to the original problem. The extension to fields involving three dependent variables presents no difficulty. Theorems of uniqueness and existence covering most types of two-dimensional flow have been established.

Flow involving three independent variables has a more nebulous character. A one-parameter family of characteristic surfaces pass through any one point in space and these envelop a conoid through the point; the lines of contact of the surfaces and the conoid, the *bicharacteristic* curves, also form a one-parameter family. There is, therefore, at first sight, a certain arbitrariness in any numerical method based on characteristics enabling one to determine the flow off a given initial surface.

Three distinct methods for three-variable problems have been proposed so far. Thornhill (1952) proposes two difference schemes, both starting from a triangle drawn on an initial surface. In the first a new point in the flow space is determined as the intersection of the three internal characteristic surfaces through the sides of the triangle. In the second the common intersection of the three internal bicharacteristics through the vertices is found.

Sauer (1950) reduces the three-variable problem to a series of two-variable problems. Working in Cartesian space he selects a sequence of equally spaced coordinate planes, x = constant, say. In each plane in turn he uses the two-dimensional method of characteristics to calculate the network intersected by families of characteristic surfaces passing through curves l (y = constant) drawn in the initial surface. Each point in this network is connected with a corresponding point in the network in an adjacent plane by a difference equation in the x-direction.

Coburn & Dolph (1949), in a paper of considerable importance, consider three-variable problems from a more formal standpoint. They draw attention to work by Titt (1939) on general non-linear hyperbolic equations in three independent variables. Titt introduces a characteristic coordinate system in terms of which the original initial value problem can be reduced to a two-dimensional problem and so establishes uniqueness and existence theorems for correctly posed three-variable problems. Coburn & Dolph suggest that any difference scheme for supersonic flow problems should be closely linked with Titt's coordinate scheme. The present author is in agreement with this view. By adopting the Titt scheme, one is sure of satisfying the requirements to ensure uniqueness and existence. At present no other scheme of equivalent soundness is known.

Coburn & Dolph have developed these ideas for the equations of steady supersonic, homentropic flow, and their characteristic coordinate system is defined as follows. A family of non-intersecting, space-like curves are drawn on the surface bearing the initial Cauchy data, and the characteristic coordinate system is based on the two families of characteristic surfaces passing through these initial curves and on the family of surfaces determined by the corresponding bicharacteristics. These three families of surfaces are taken as coordinate surfaces, and the equation of potential flow is replaced by two characteristic equations and symmetry conditions. The characteristic system of equations is in a form suitable for solution by a method of finite differences but Coburn & Dolph do not consider the details of this.

Of the methods proposed for practical application Sauer's is the closest to the formal approach of Coburn & Dolph, since he employs two families of characteristic surfaces and a third non-characteristic family. However, Sauer chooses the third family to be planes x = constant, and in so doing is unable to satisfy Coburn & Dolph's requirement that two coordinate directions shall be bicharacteristic. When this condition is satisfied the third family of coordinate surfaces is determined automatically. When Coburn & Dolph's method is applied, in general, none of the coordinate curves in the characteristic system are known in advance and they have to be determined, step by step, together with the physical variables arising in the problem in question.

In the present paper Coburn & Dolph's work is carried a stage further. Firstly, their method is generalized to take account of motion with entropy changes and vorticity (viscosity and heat conduction are neglected). Even the simplest three-variable problems involve shocks of variable curvature, so this step is essential. The basic characteristic coordinate system is set up exactly in the manner proposed by Coburn & Dolph. When the motion is referred to this system two of the transformed equations are generalizations of the second characteristic equations familiar in non-homentropic problems in two variables. A third equation relates derivatives of pressure and velocity components. The fourth equation, the energy equation, directly connects total speed with variables of state. The entropy does not enter these equations explicitly, and appears only in the equation of state and the condition that entropy remains constant in the stream direction.

A finite difference scheme is considered for the numerical solution of the equations in practical cases. The basis of the scheme is the construction of a sequence of linear characteristic networks in the surfaces determined by the bicharacteristics. The method of construction of any one network is similar to that adopted in non-homentropic flow in two dimensions. An extra difference equation connects any two adjacent networks.

In the general case, when derivatives with respect to all space variables occur non-linearly, the calculation of the coordinate system is completely interwoven with that of the dependent variables themselves, and the computation required is indeed formidable. However, if we know the solution to a certain non-linear flow problem, the equations governing a field of flow which is a linear perturbation of this are linear, and can be expressed in terms of a characteristic coordinate system defined by the basic flow. Such perturbation problems are of wide practical interest and have been examined previously, chiefly by Ferri (1952), but also by Ferrari (1936) and Guderley (1947).

Ferri has developed a Linearized Method of Characteristics to apply to a number of three-variable problems of this simplified type. He considers three-dimensional linear perturbations of known two-dimensional fields of flow. Among the applications he considers are axially symmetrical flow past bodies of revolution which differ slightly from conical shape, flow past bodies of revolution (not necessarily thin) at small angles of yaw, and fields of flow in which there is a slight departure from plane flow conditions. In all cases his perturbation equations are referred to the characteristic network of the basic flow. In the case of plane flow the third independent variable is taken to be the Cartesian coordinate normal to the flow plane. In the axially symmetrical case, the angle of rotation of a basic meridian plane is used.

Essentially, in Ferri's Linearized Method of Characteristics, the departure point is some appropriate two-dimensional method. Here we shall arrive at a Linearized Method of Characteristics from a different approach. We shall first establish the equation governing the perturbations of a basic *three*-dimensional flow referred to Coburn & Dolph's system of characteristic coordinates. We shall then consider how these equations simplify when the basic flow degenerates into a plane or axially symmetrical flow. In this way it is easier to make sure that Coburn & Dolph's system of equations is retained in the form accepted for fully three-dimensional flow.

It is difficult to compare the equations obtained here with Ferri's equations, since not only are the two approaches different but, also, the dependent variables are handled in a different manner. However, in the special cases treated the characteristic coordinate systems are the same and it should be possible to deduce one form of equations from the other.

2. The equations of rotational motion in a generalized coordinate system

Let the position of a point in a field of steady, supersonic rotational flow be defined by the coordinate vector x^i with metric tensor g_{ij}, and let u^i be the velocity vector at this point. Denote the pressure, density and entropy by p, ρ and S respectively. We shall neglect viscosity, heat conduction and radiation.

Referred to this coordinate system the Eulerian equations of motion are

$$u^j u_{i,j} + \frac{1}{\rho} p_{,i} = 0, \tag{2.1}$$

together with the equation of continuity,

$$g^{jk} u_{j,k} + \frac{u^k}{\rho} \rho_{,k} = 0. \tag{2.2}$$

In steady flow there is no variation in entropy in the stream direction, so that

$$u^j S_{,j} = 0. \tag{2.3}$$

The equation of state may be written

$$p = p(\rho, S). \tag{2.4}$$

From (2.4),

$$p_{,i} = \left(\frac{\partial p}{\partial \rho}\right) \rho_{,i} + \left(\frac{\partial p}{\partial S}\right) S_{,i}. \tag{2.5}$$

We write

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right). \tag{2.6}$$

Then, from (2.3) and (2.5), we obtain the relation

$$u^j p_{,j} = c^2 u^j \rho_{,j}. \tag{2.7}$$

Using this, (2.2) may be written

$$g^{jk} u_{j,k} + \frac{u^k}{\rho c^2} p_{,k} = 0. \tag{2.8}$$

In supersonic flow real characteristic surfaces are associated with (2.1), (2.8), (2.3) and (2.4). Through each point x^i pass a single-parameter family of surfaces which are characteristic for the velocity vector, the pressure and the density. In addition, each surface through x^i which contains the stream direction is characteristic for the entropy.

Consider first (2.1) and (2.8). Then the surface Σ,

$$z(x^i) = \text{constant},$$

is characteristic if, when we specify on Σ values of u^i, p and ρ, equations (2.1) and (2.8) admit solutions with arbitrary values of derivatives of these quantities leading out of the surface. An elementary calculation shows that if λ^i is the unit vector normal to Σ at x^i, then the surface is characteristic if

$$(u^i\lambda_i)^2 = c^2. \tag{2.9}$$

This means that there are ∞^1 characteristic surfaces through λ^i, which are tangential to a cone, the characteristic cone, with axis along the velocity direction. The component of velocity normal to a characteristic surface is equal to the local speed of sound. We shall be interested only in the nappe of the cone downstream of x^i, on which

$$u^i\lambda_i = +c. \tag{2.10}$$

We observe that these characteristic surfaces are defined exactly as in homentropic flow, so we may make use of the properties of such surfaces deduced by Coburn & Dolph.

Now consider (2.3) and write

$$S_{,j} = \lambda_j s + \mu_j t + \nu_j u,$$

where λ_j is a unit vector normal to the surface and μ_j, ν_j are unit vectors lying in the surface

$$z(x^i) = \text{constant}.$$

Then we find

$$u^j\lambda_j s + u^j\mu_j t + u^j\nu_j u = 0,$$

and we are therefore unable to determine s uniquely if

$$u^j\lambda_j = 0.$$

It follows that surfaces containing the velocity vector are characteristic for the entropy.

3. Equations of motion in characteristic form

To obtain the characteristic equations corresponding to (2.1) and (2.8) we may employ the same coordinate system as that defined by Coburn & Dolph for homentropic flow.

In Coburn & Dolph (1949), initial data (in the general case, values of u^i, p, ρ, S compatible with the equation of state) are given on some non-characteristic initial surface. On this surface an ∞^1-family of curves are drawn which are space-like with respect to the local Mach cone. Through each curve of this family pass two surfaces which are characteristic in relation to (2.1) and (2.8). In this way two families of characteristic surfaces are defined in the region of flow and these will intersect in an ∞^2-family of curves, which will contain the original ∞^1-family on the initial surface.

At each point on a curve of this ∞^2-family two bicharacteristic directions are defined. These are the directions of the lines of contact between the two characteristic surfaces and the characteristic conoid at the point. Thus at each point in the flow region three directions are defined. On these Coburn & Dolph base their characteristic coordinate system.

Let l^i be the unit vector along a line of the ∞^2-family. Let λ^i, λ'^i be unit vectors at this point directed along normals to the two characteristic surfaces through the point, and let t^i, t'^i be the corresponding bicharacteristic directions at this point. The required coordinate system is to be based on the triad l^i, t^i, t'^i.

Following Coburn & Dolph we now transform (2.1) and (2.8) so that they are referred to the coordinate triad l^i, t^i, t'^i. In particular, we require that only derivatives in the new coordinate directions shall appear in the equations of motion.

We express $u_{i,j}$ as the sum of components along λ^i, t^i, l^i. Then

$$u_{i,j} = \lambda_j a_i + t_j b_i + l_j c_i$$

and

$$u^j u_{i,j} = u^j \lambda_j a_i + u^j t_j b_i + u^j l_j c_i.$$

Equation (2.1) becomes

$$u^j \lambda_j a_i + u^j t_j b_i + u^j l_j c_i + \frac{1}{\rho} p_{,i} = 0. \tag{3.1}$$

Equation (2.8) becomes

$$g^{jk}(\lambda_k a_j + t_k b_j + l_k c_j) + \frac{1}{\rho c^2} u^k p_{,k} = 0,$$

or

$$\lambda^j a_j + t^j b_j + l^j c_j + \frac{1}{\rho c^2} u^k p_{,k} = 0. \tag{3.2}$$

Take the scalar product of λ^i with (3.1). Then

$$(u^j \lambda_j) \lambda^i a_i + u^j b_j \lambda^i b_i + u^j l_j \lambda^i c_i + \frac{1}{\rho} \lambda^i p_{,i} = 0. \tag{3.3}$$

Multiply (3.2) by c and subtract from (3.3). Then, using the property $\lambda^i u_i = +c$ (λ^i is the outward normal on the downstream Mach cone), we find

$$(u^j t_j \lambda^i - c t^i) b_i + (u^j l_j \lambda^i - c l^i) c_i + \frac{1}{\rho c}(c\lambda^i - u^i) p_{,j} = 0. \tag{3.4}$$

If q is the total speed we may write (see Coburn & Dolph, equation (2.6))

$$u^i = c\lambda^i + \sqrt{(q^2 - c^2)}\, t^i, \tag{3.5}$$

and (3.4) reduces to

$$(u^j t_j \lambda^i - c t^i) b_i + (u^j l_j \lambda^i - c l^i) c_i - \frac{\sqrt{(q^2 - c^2)}}{\rho c} t^i p_{,i} = 0. \tag{3.6}$$

We now evaluate b_i and c_i. We find

$$b_i + t^j l_j c_i = t^j u_{i,j},$$

$$l^j t_j b_i + c_i = l^j u_{i,j},$$

so if

$$l^j t_j = \cos\psi = a, \tag{3.7}$$

$$b_i(1 - \cos^2\psi) = t^j u_{i,j} - a l^j u_{i,j},$$

$$c_i(1 - \cos^2\psi) = l^j u_{i,j} - a t^j u_{i,j},$$

and (3.6) reduces to

$$\frac{1}{1-a^2}\{u^j(t_j - a l_j)\lambda^i - c(t^i - a l^i)\} t^k u_{i,k} +$$

$$+ \frac{1}{1-a^2}\{u^j(l_j - a t_j)\lambda^i - c(l^i - a t^i)\} l^k u_{i,k} - \frac{\sqrt{(q^2 - c^2)}}{\rho c} t^k p_{,k} = 0. \tag{3.8}$$

Similarly, starting from the resolution

$$u_{i,j} = \lambda'_j a'_i + t'_j b'_i + l_j c'_i,$$

we can derive the equation

$$\frac{1}{1-a^2}\{u^j(t'_j - al_j)\lambda'^i - c(t'^i - al^i)\}\, t'^k u_{i,k} + \\ + \frac{1}{1-a^2}\{u^j(l_j - at'_j)\lambda'^i - c(l^i - at'^i)\}\, l^k u_{i,k} - \frac{\sqrt{(q^2-c^2)}}{\rho c}\, t'^k p_{,k} = 0. \quad (3.9)$$

Equations (3.8) and (3.9) are the two 'second characteristic conditions'. They are generalizations of Coburn & Dolph's equations and express the conditions that derivatives of u^i or p (or ρ) leading out of the characteristic surface exist, even though they are not determined uniquely. Equation (3.8) contains only derivatives along t^i and l^i, and equation (3.9) contains only derivatives along t'^i and l^i.

We require three further equations to determine u^i, p, ρ, S in the characteristic coordinate system. The first is a further relation between derivatives in the t^i, t'^i and l^i directions.

Equation (2.18) of Coburn & Dolph (1949) applies here and is

$$u^i = \frac{c^2}{(1-d)\sqrt{(q^2-c^2)}}(t^i + t'^i) - \frac{ac^2}{(1-a^2)\sqrt{(q^2-c^2)}}\, l^i, \quad (3.10)$$

where

$$d = \cos\phi = t^i t'_i. \quad (3.11)$$

Substitute the expression (3.10) for u^j in (2.1), but leave $u_{i,j}$ unchanged, and take the scalar product of l^i with the resulting equation. In this way we find

$$\frac{c^2}{(1-d)\sqrt{(q^2-c^2)}}\, l^i(t^j u_{i,j} + t'^j u_{i,j}) - \\ - \frac{ac^2}{(1-a^2)\sqrt{(q^2-c^2)}}\, l^i l^j u_{i,j} + \frac{1}{\rho} l^j p_{,j} = 0. \quad (3.12)$$

In most practical applications the flow originates from a region of constant total energy and the relation

$$q^2 + \int \frac{dp}{\rho} = \text{constant} \quad (3.13)$$

is true throughout the fluid. The constant on the right has the same value throughout the flow field irrespective of the presence of shock waves.

Equation (3.13) provides the second relation required. The third relation involves the entropy.

Suppose that ν^i is the unit vector normal to the surface determined by l^i and the velocity vector u^i. Let m^i be the direction intersected by this surface on the surface determined by the bicharacteristics t^i and t'^i. Write

$$S_{,i} = \nu_i A + m_i B + l_i C.$$

Then

$$B + m^i l_i C = m^i S_{,i},$$

$$l^i m_i B + C = l^i S_{,i},$$

and if

$$e = l^i m_i,$$

$$B = \frac{1}{1-e^2}(m^i S_{,i} - e l^i S_{,i}),$$

$$C = \frac{1}{1-e^2}(l^i S_{,i} - e m^i S_{,i}).$$

Hence, from (2.3),

$$u^j m_j B + u^j l_j C = 0,$$

or

$$(u^j m_j - e u^j l_j) m^i S_{,i} + (u^j l_j - e u^j m_j) l^i S_{,i} = 0. \tag{3.14}$$

The finite difference scheme

The equations of motion referred to characteristic coordinates, (3.8), (3.9), (3.12), (3.13) and (3.14), are in the form most suitable for solution by numerical methods and we now put forward a finite difference scheme. The basis of this is the construction of a sequence of linear characteristic networks in the surfaces determined by the bicharacteristics. Three difference equations are employed in each surface and an additional difference equation connects the point to be determined with the corresponding point in the adjacent surface, where conditions are already known.

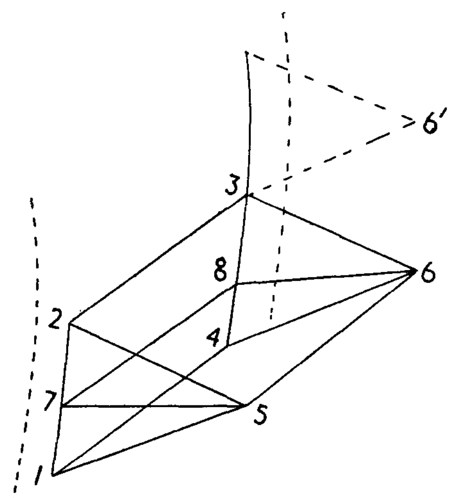

Figure 1. The cell of the characteristic network.

The cell in the network is shown in figure 1. We start from the quadrangle 1234 on a space-like initial surface. The segments 23 and 14 lie on initial lines (l^i) on the surface. The segments 12 and 43 lie on surfaces determined by bicharacteristics on the initial surface.

It is assumed that the characteristic network in the surface through 12 is known. (The first surface found may be in a plane of symmetry.)

Segments 75 and 86 are the intersections of the surface through l^i and u^i on the elements 346, 215.

Equation (3.8) is applied along the segment 46. The values of $t^k u_{i,k}$ are taken to be those at point 4, and $t^k u_{i,k}$, $t^k p_{,k}$ are replaced by difference ratios

$$\frac{\Delta u_i}{\text{distance } 46}, \quad \frac{\Delta p}{\text{distance } 46}.$$

Equation (3.9) is used in a similar manner to set up a difference equation along the segment 36. Equation (3.14) is used to determine the change in S along 86, that is, to find S at 6. Equation (3.12) is used to set up a difference equation along 56; the values of $l^i t^j u_{i,j}$ and $l^i t'^j u_{i,j}$ are taken to be those at point 5. Equation (3.13) completes the system, which can then be solved by an iterative procedure. When 6 has been found the point 6′ is next determined, and so on.

The process must begin in a surface where conditions are known, such as a plane of symmetry.

In general, it appears that the characteristic coordinate system is completely non-orthogonal. Coburn & Dolph made the conjecture that the diredtion l^i is normal to t^i and t'^i throughout the flow space provided that this is the case on the original initial surface. They did not prove the conjecture and in spite of a thorough investigation the present author can find no evidence of its validity. However, the obliqueness of the coordinate system does not add seriously to the formidable amount of numerical work inherent in non-linear problems in three dimensions.

4. Linear perturbations of steady supersonic flow problems

At the present stage, when so little is known about three-dimensional flow problems, the need for calculating completely non-linear fields of flow is limited. To begin with, an approximate technique is required which, by introducing some simplification in the general method, leads fairly quickly to an assessment of three-dimensional effects. This is the motive behind the Linearized Method of Characteristics developed by Ferri and, in more limited forms, by Ferrari and Guderley.

We now introduce a Linearized Method of Characteristics from a new and more general point of view, starting from the characteristic equations set out in § 3. Ferri considers the linear, three-dimensional perturbation of certain basic non-linear flows involving two independent variables, such as plane or axially symmetrical flow past non-slender bodies. Here we shall first examine linear perturbations of a general three-dimensional flow and then treat perturbations of flow in two variables as special cases. By this means it is much easier to ensure that the process of linearization does not distort the characteristic coordinate scheme described in § 3.

We start with a field of flow in which, at every point x^i, the velocity vector U^i, pressure P, density R, and entropy S are assumed to be given. We then consider a field of flow which is a linear perturbation of this given basic field with velocity vector $U^i + u^i$, pressure $P + p$, density $R + \rho$, entropy $S + s$. We assume that the dependent variables in the additional flow field, that is, the flow field remaining when the given field is subtracted from the perturbed flow field, are small quantities of the first order, whose squares and products can be neglected. We refer both the perturbed and the basic fields of flow to the same system of coordinates x^i, with metric tensor g_{ij}.

The equations governing the perturbed field are

$$(U^j+u^j)(U_i+u_i)_{,j}+\frac{1}{R+\rho}(P+p)_{,i} = 0, \tag{4.1}$$

$$g^{jk}(U_j+u_j)_{,k}+\frac{(U^k+u^k)}{(R+\rho)(C+c)^2}(P+p)_{,k} = 0, \tag{4.2}$$

$$(U^j+u^j)(S+s)_{,j} = 0, \tag{4.3}$$

$$P+p = f(R+\rho, S+s). \tag{4.4}$$

In the basic field the same equations are satisfied with all terms in u^i, p, ρ, s omitted. If we subtract these equations from the corresponding equations (4.1) to (4.4) and retain only terms of the first order in u^i, p, ρ and s we obtain the following equations to determine the additional flow field,

$$U^ju_{i,j}+\frac{1}{R}p_{,i}+u^jU_{i,j}-\frac{\rho}{R^2}P_{,i} = 0, \tag{4.5}$$

$$g^{jk}u_{j,k}+\frac{1}{RC^2}U^kP_{,k}+\frac{1}{RC^2}\left\{u^kP_{,k}-\left(\frac{\rho}{R}+\frac{2c}{C}\right)U^kP_{,k}\right\} = 0, \tag{4.6}$$

$$U^js_{,j}+u^jS_{,j} = 0, \tag{4.7}$$

$$p = \rho\frac{\partial P}{\partial R}+s\frac{\partial P}{\partial S}. \tag{4.8}$$

The coefficients of the derivatives of u^i, p, and s in (4.5), (4.6) and (4.7) are the same as the corresponding coefficients in the equations of the basic flow. It follows that the characteristic properties of the additional flow are identical with those of the basic flow, so that the characteristic equations of the additional flow can be referred to the coordinate system defined in § 3. When this is done (4.5), (4.6) are replaced by the system

$$\begin{aligned}&\frac{1}{(1-a^2)}\{(U^jt_j\lambda^i-Ct^i)-a(U^jl_j\lambda^i-Cl^i)\}t^ju_{i,j}+\\&\quad+\frac{1}{(1-a^2)}\{(U^jl_j\lambda^i-Cl^i)-a(U^jt_j\lambda^i-Ct^i)\}l^ju_{i,j}-\\&-\frac{1}{RC}\sqrt{(Q^2-C^2)}t^ip_{,i}-\frac{1}{RC}\left\{u^kP_{,k}-\left(\frac{\rho}{R}+\frac{2c}{C}\right)U^kP_{,k}\right\}-\\&\qquad-\frac{\rho}{R^2}\lambda^iP_{,i}+u^j\lambda^iU_{i,j} = 0,\end{aligned} \tag{4.9}$$

$$\begin{aligned}&\frac{1}{1-a^2}\{(U^jt'_j\lambda'^i-Ct'^i)-a(U^jl_j\lambda'^i-Cl^i)\}t'^ju_{i,j}+\\&\quad+\frac{1}{(1-a^2)}\{(U^jl_j\lambda'^i-Cl^i)-a(U^jt'_j\lambda'^i-Ct'^i)\}l^ju_{i,j}-\\&-\frac{1}{RC}\sqrt{(Q^2-C^2)}t'^ip_{,i}-\frac{1}{RC}\left\{u^kP_{,k}-\left(\frac{\rho}{R}+\frac{2c}{C}\right)U^kP_{,k}\right\}-\\&\qquad-\frac{\rho}{R^2}\lambda'^iP_{,i}+u^j\lambda'^iU_{i,j} = 0,\end{aligned} \tag{4.10}$$

$$\frac{l^i C^2}{(1-d)\surd(Q^2-C^2)}(t^j u_{i,j}+t'^j u_{i,j}) - \frac{aC^2}{(1-a^2)\surd(Q^2-C^2)} l^i l^j u_{i,j} + \\ + l^i u^j U_{i,j} - \frac{\rho}{R^2} l^i P_{,i} + \frac{1}{R} l^i p_{,i} = 0, \quad (4.11)$$

$$qQ + \int \frac{dp}{R} - \int \frac{\rho\, dP}{R^2} = 0. \quad (4.12)$$

Equations (4.9) and (4.10) are the 'second characteristic conditions'. Equation (4.11) corresponds to equation (3.12) for the basic flow, and (4.12) is derived from the energy equation. These are to be solved in conjunction with (4.7) and (4.8).

Simplification when basic flow involves two independent variables

The equations simplify for basic fields which involve only two independent variables, for example, plane flow, flow with axial symmetry and conical flow. In all such fields the basic flow is identical in each of a one-parameter family of surfaces and it is possible to choose the vectors l^i to be normal everywhere to the surfaces of flow. With this choice l^j is normal to U^j, t^j, t'^j, λ^j and λ'^j and $a = 0$. Equations (4.9), (4.10) and (4.11) then simplify to

$$(U^j t_j \lambda^i - Ct^i) t^j u_{i,j} - Cl^i l^j u_{i,j} - \frac{1}{RC}\surd(Q^2-C^2) t^i p_{,i} - \\ - \frac{1}{RC}\left\{u^k P_{,k} - \left(\frac{\rho}{R}+\frac{2c}{C}\right)U^k P_{,k}\right\} - \frac{\rho}{R^2}\lambda^i P_{,i} + u^j \lambda^i U_{i,j} = 0, \quad (4.13)$$

$$(U^j t'_j \lambda^i - Ct'^i) t'^j u_{i,j} - Cl^i l^j u_{i,j} - \frac{1}{RC}\surd(Q^2-C^2) t'^i p_{,i} - \\ - \frac{1}{RC}\left\{u^k P_{,k} - \left(\frac{\rho}{R}+\frac{2c}{C}\right)U^k P_{,k}\right\} - \frac{\rho}{R^2}\lambda'^i P_{,i} + u^j \lambda'^i U_{i,j} = 0, \quad (4.14)$$

$$\frac{l^i C^2}{(1-d)\surd(Q^2-C^2)}(t^j u_{i,j}+t'^j u_{i,j}) + l^i u^j U_{i,j} - \frac{\rho}{R^2} l^i P_{,i} + \frac{1}{R} l^i p_{,i} = 0. \quad (4.15)$$

We now develop these equations further in the two special cases of basic plane flow and basic axially symmetrical flow.

Linear perturbations of plane flow

In the plane flow case we take l^i in the direction normal to the flow plane with corresponding coordinate z. The characteristic surfaces are then cylindrical surfaces and can be written

$$\alpha = \text{constant}, \qquad \beta = \text{constant},$$

with generators parallel to the z-direction. Let h_α, h_β be length parameters along the coordinate curves in the α- and β-directions respectively. Then if μ is the Mach angle of the basic flow, the original coordinate vector x^i is Cartesian, with x^1 and x^2 in the basic flow plane and x^3 normal to the flow plane. We then find that

$$\begin{aligned} ds^2 &= dx_1^2+dx_2^2+dx_3^2 \\ &= h_\alpha^2\,d\alpha^2+h_\beta^2\,d\beta^2+2h_\alpha h_\beta\cos 2\mu\,d\alpha d\beta+dz^2. \end{aligned} \tag{4.16}$$

The coefficients in (4.16) are related to corresponding coefficients defined in § 3 of Coburn & Dolph's paper by

$$A = h_\alpha, \qquad B = h_\beta, \qquad C = 1, \qquad E = h_\alpha h_\beta \cos 2\mu.$$

Since x^i is Cartesian, the directional derivatives of u^i become, simply,

$$t^j u_{i,j} = \frac{1}{h_\beta}\frac{\partial u_i}{\partial \beta}, \qquad t'^j u_{i,j} = \frac{1}{h_\alpha}\frac{\partial u_i}{\partial \alpha}, \qquad l^j u_{i,j} = \frac{\partial u_i}{\partial z}.$$

If θ is the angle between the basic velocity direction and the x-axis, the unit vectors associated with the coordinate system and the velocity vector are defined as follows:

$$\begin{aligned} t^i &= \{\cos(\theta-\mu), \sin(\theta-\mu), 0\}, \\ t'^i &= \{\cos(\theta+\mu), \sin(\theta+\mu), 0\}, \\ l^i &= \{0, 0, 1\}, \\ \lambda^i &= \{-\sin(\theta-\mu),\ +\cos(\theta-\mu), 0\}, \\ \lambda'^i &= \{+\sin(\theta+\mu),\ -\cos(\theta+\mu), 0\}, \\ U^j &= \{Q\cos\theta, Q\sin\theta, 0\}. \end{aligned}$$

Making use of these relations, (4.13), (4.14) and (4.15) are transformed, after some reduction, to the equations

$$-Q\sin\theta\frac{\partial u_1}{h_\beta\,\partial\beta}+Q\cos\theta\frac{\partial u_2}{h_\beta\,\partial\beta}-\frac{\cot\mu}{R}\frac{\partial p}{h_\beta\,\partial\beta}-C\frac{\partial u_3}{\partial z}+K+L=0, \tag{4.17}$$

$$Q\sin\theta\frac{\partial u_1}{h_\alpha\,\partial\alpha}-Q\cos\theta\frac{\partial u_2}{h_\alpha\,\partial\alpha}-\frac{\cot\mu}{R}\frac{\partial p}{h_\alpha\,\partial\alpha}-C\frac{\partial u_3}{\partial z}+K+M=0, \tag{4.18}$$

where

$$\begin{aligned} K = &-\frac{\operatorname{cosec}2\mu}{RC}\Big[\{u^1\sin(\theta+\mu)-u^2\cos(\theta+\mu)\}\frac{\partial P}{h_\beta\,\partial\beta}+ \\ &+\{-u^1\sin(\theta-\mu)+u^2\cos(\theta-\mu)\}\frac{\partial P}{h_\alpha\,\partial\alpha}\Big]+ \\ &+\frac{c}{RQ\sin^2\mu\cos\mu}\left(\frac{\partial P}{h_\beta\,\partial\beta}+\frac{\partial P}{h_\alpha\,\partial\alpha}\right), \end{aligned} \tag{4.19}$$

$$\begin{aligned} L = &\frac{\rho}{R^2}\cot\mu\frac{\partial P}{h_\beta\partial\beta}+\operatorname{cosec}2\mu\{u^1\sin(\theta+\mu)-u^2\cos(\theta+\mu)\}\times \\ &\times\left\{-\sin(\theta-\mu)\frac{\partial U_1}{h_\beta\partial\beta}+\cos(\theta-\mu)\frac{\partial U_2}{h_\beta\partial\beta}\right\}+\operatorname{cosec}2\mu\{-u^1\sin(\theta-\mu)- \\ &-u^2\cos(\theta-\mu)\}\left\{-\sin(\theta-\mu)\frac{\partial U_1}{h_\alpha\,\partial\alpha}+\cos(\theta-\mu)\frac{\partial U_2}{h_\alpha\,\partial\alpha}\right\}, \end{aligned} \tag{4.20}$$

$$M = \frac{\rho}{R^2}\cot\mu\frac{\partial P}{h_\alpha\,\partial\alpha}+\operatorname{cosec}2\mu\{u^1\sin(\theta+\mu)-u^2\cos(\theta+\mu)\}\times$$
$$\times\left\{\sin(\theta+\mu)\frac{\partial U_1}{h_\beta\,\partial\beta}-\cos(\theta+\mu)\frac{\partial U_2}{h_\beta\,\partial\beta}\right\}+\operatorname{cosec}2\mu\{-u^1\sin(\theta-\mu)+$$
$$+u^2\cos(\theta-\mu)\}\left\{\sin(\theta+\mu)\frac{\partial U_1}{h_\alpha\,\partial\alpha}-\cos(\theta+\mu)\frac{\partial U_2}{h_\alpha\,\partial\alpha}\right\}, \quad (4.21)$$

and

$$\tfrac{1}{2}Q\sec\mu\left(\frac{\partial U_3}{h_\beta\,\partial\beta}+\frac{\partial U_3}{h_\alpha\,\partial\alpha}\right)+\frac{1}{R}\frac{\partial p}{\partial z} = 0. \quad (4.22)$$

The system is completed by the perturbed forms of the energy equations, the equation of state and the condition of conservation of entropy in the stream direction. These give

$$u_1U_1+u_2U_2-\int\frac{\rho\,dP}{R^2}+\int\frac{dp}{R} = 0, \quad (4.23)$$

$$p = \rho\frac{\partial P}{\partial R}+s\frac{\partial P}{\partial S}, \quad (4.24)$$

$$u^jS_{,j}+Qm^js_{,j} = 0, \quad (4.25)$$

where m^j is a unit vector in the basic stream direction.

Linear perturbations of flow with axial symmetry

When the basic flow has axial symmetry it is convenient to refer the basic field to cylindrical polars x^i with the x^1-axis along the axis of symmetry. To calculate the perturbed field we use characteristic surfaces

$$\alpha = \text{constant}, \qquad \beta = \text{constant},$$

obtained by rotating the characteristic curves in the basic flow plane about the axis of symmetry. Corresponding length parameters h_α, h_β along these curves are introduced. The meridian planes are used as the third set of surfaces, so the third coordinate,

$$x^3 = \phi,$$

is the angle between any meridian plane and a fixed meridian plane. In terms of the usual notation for cylindrical polars,

$$x^1 = z, \qquad x^2 = r.$$

Since the original coordinate space is not Cartesian the directional derivatives are not as easy to calculate as in the case of basic plane flow.

Let y^i be the coordinate vector in the characteristic space with metric tensor G_{ij}. Then

$$y^1 = \beta, \qquad y^2 = \alpha, \qquad y^3 = x^3 = \phi,$$
$$ds^2 = g_{mn}\,dx^m\,dx^n = G_{mn}\,dy^m\,dy^n,$$

where

$$g_{mn} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (x^2)^2 \end{bmatrix},$$

and a short calculation shows that

$$G_{mn} = \begin{bmatrix} h_\beta^2 & h_\alpha h_\beta \cos 2\mu & 0 \\ h_\alpha h_\beta \cos 2\mu & h_\alpha^2 & 0 \\ 0 & 0 & (x^2)^2 \end{bmatrix}$$

The unit vectors in the characteristic coordinate triad are

$$t^i = \{\cos(\theta-\mu), \sin(\theta-\mu), 0\},$$
$$t'^i = \{\cos(\theta+\mu), \sin(\theta+\mu), 0\},$$
$$l^i = \{0, 0, 1/x^2\}.$$

The covariant derivative of u_i is

$$u_{i,j} = \frac{\partial u_i}{\partial x^j} - \begin{Bmatrix} m \\ ij \end{Bmatrix} u_m,$$

where $\begin{Bmatrix} m \\ ij \end{Bmatrix}$ is the Christoffel symbol. In basic plane flow all components of $\begin{Bmatrix} m \\ ij \end{Bmatrix}$ are zero but in the present case three components do not vanish. They are

$$\begin{Bmatrix} 2 \\ 33 \end{Bmatrix} = -x^2, \qquad \begin{Bmatrix} 3 \\ 23 \end{Bmatrix} = 1/x^2, \qquad \begin{Bmatrix} 3 \\ 32 \end{Bmatrix} = 1/x^2.$$

As a result additional terms are introduced into the expression for the directional derivatives along l^i, t^i, t'^i.

The equations governing the perturbed motion are finally found to be as follows:

$$-Q\sin\theta\frac{\partial u_1}{h_\beta\,\partial\beta} + Q\cos\theta\frac{\partial u_2}{h_\beta\,\partial\beta} - \frac{C\partial u_3}{r^2\partial\phi} - \frac{Cu_2}{r} - \frac{\cot\mu}{R}\frac{\partial p}{h_\beta\,\partial\beta} + K + L = 0, \tag{4.26}$$

$$Q\sin\theta\frac{\partial u_1}{h_\alpha\,\partial\alpha} - Q\cos\theta\frac{\partial u_2}{h_\alpha\,\partial\alpha} - \frac{C\partial u_3}{r^2\partial\phi} - \frac{Cu_2}{r} - \frac{\cot\mu}{R}\frac{\partial p}{h_\alpha\,\partial\alpha} + K + M = 0, \tag{4.27}$$

$$\tfrac{1}{2}Q\sec\mu\left\{\frac{1}{r}\left(\frac{\partial u_3}{h_\alpha\,\partial\alpha} + \frac{\partial u_3}{h_\beta\,\partial\beta}\right)\right\} + \frac{1}{R}\frac{\partial p}{r\partial\phi} = 0, \tag{4.28}$$

together with (4.10), (4.24) and (4.25). In (4.26) and (4.27), K, L and M are the expressions defined in (4.19), (4.20) and (4.21).

Applications

Ferri has considered a number of practical problems to which his linearized method of characteristics may be applied. In the case of perturbed plane flow he takes a finite wing, in any cross-section of which the flow is substantially two-dimensional. As an example of perturbed axially symmetrical flow he considers bodies of revolution at small angles of yaw and non-symmetrical conical bodies. Both classes of problems are important and merit futher attention. We shall describe briefly how they may be attacked by the present linearized characteristics method.

In both applications we must first apply the method of characteristics for two independent variables to calculate the basic flow. We then know the characteristic coordinate network for the additional flow and at each point of this we can tabulate the functions K and L and the coefficients of derivatives of additional flow variables. The additional flow can then be calculated by integrating the three-dimensional equations along the basic network. The resulting equations are all linear and the numerical process of solution is therefore comparatively simple, although the boundary condition may introduce difficulties. Conditions must be satisfied partly on a shock wave and partly on the body surface. It is necessary to construct the change in shape of shock due to third-dimensional effects as the calculation proceeds.

Simplifications can be introduced into many problems concerning perturbations of plane flow. To the first approximation, plane supersonic flow past a curved aerofoil can be treated as that through a simple wave behind an attached curved shock with negligible entropy variation. A three-dimensional perturbation of this can be treated as a linear non-homentropic perturbation of a simple wave. This is governed by equations (4.17) to (4.25), where now all the derivatives of the basic variables along curves of one characteristic family vanish.

A similar simplification arises when three-dimensional perturbations of two-dimensional wind tunnel flow are examined. Part of the basic flow is again a simple wave and it is not difficult to extend the technique of Meyer & Holt (1951) to correct wind tunnels for three-dimensional departures from specified shape.

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